



Trotter products and reaction–diffusion equations

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ARTICLE INFO

Article history:

Received 26 December 2007

Dedicated to Professor Jesús S. Dehesa on the occasion of his 60th birthday

MSC:

26A33

47G30

47D07

31B10

Keywords:

Reaction–diffusion equation

Product formula

Pseudodifferential operator

Feller semigroup

ABSTRACT

In this paper, we study a class of generalized diffusion–reaction equations of the form $\frac{\partial u}{\partial t}(x, t) = (Au(\cdot, t))(x) + f(x, u(x, t))$, where A is a pseudodifferential operator which generates a Feller semigroup. Using the Trotter product formula we give a corresponding discrete time integro–difference equation for numerical solutions.

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1. Introduction

Reaction–diffusion equations play a very important role in many areas of science and engineering as physics, biology, hydrogeology, porous media etc. The classical one-dimensional reaction–diffusion equation is

$$\frac{\partial u}{\partial t}(x, t) = C \frac{\partial^2 u}{\partial x^2}(x, t) + f(u(x, t)), \quad u(x, 0) = u_0(x),$$

where $u(x, t)$ depends on location x and time t . This equation is inadequate to model many real situations, named anomalous diffusions, when the growth rate or the shape of the particle distribution is different than the classical model predicts (see [1]). These models are described by equations that contain fractional derivatives in the place of the usual integer order derivatives. A known model for an anomalous diffusion is the fractional diffusion equation, where the usual second derivative in space is replaced by a fractional derivative of order α , $0 < \alpha < 2$,

$$\frac{\partial u}{\partial t}(x, t) = C \frac{\partial^\alpha u}{\partial x^\alpha}(x, t) + f(u(x, t)), \quad u(x, 0) = u_0(x).$$

We observe that $\frac{\partial^\alpha}{\partial x^\alpha}$ is a pseudodifferential operator. In this paper, starting from fractional diffusions, we analyze the generalized reaction–diffusion equation

$$\frac{\partial u}{\partial t}(x, t) = (Au(\cdot, t))(x) + f(u(x, t)), \quad u(x, 0) = u_0(x),$$

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where A is a pseudodifferential operator which generates a strongly continuous positive contraction semigroup (called Feller semigroup). In [2] this equation is considered with A as a pseudodifferential operator which generates a convolution semigroup (in this case A is an operator “with constant coefficients”). We will study the same equation, but A will be a pseudodifferential operator which generates a Feller semigroup (the symbol of A will depend of x , A will be an operator “with variable coefficients”). We will follow the procedures of [2], but an important tool in this extension will be the integral representation of the operators which form the Feller semigroup (see [3]).

2. Abstract reaction–diffusion equation. Trotter product formula

Let $(X, \|\cdot\|)$ be a Banach space and $\{T(t)\}_{t \geq 0}$ a strongly continuous semigroup on X (for any $t > 0$, $T(t) : X \rightarrow X$ is a linear operator and there exists $M > 0$ such that $\|T(t)x\| \leq M\|x\|$; $T(0) = I$; $T(t+s) = T(t)T(s)$ for $t, s \geq 0$; $t \rightarrow T(t)x$ is continuous in the norm $\|\cdot\|$, for all $x \in X$). We denote by $(A, D(A))$ the generator of the semigroup $\{T(t)\}_{t \geq 0}$,

$$D(A) = \left\{ x \in X \mid \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t} \text{ exists} \right\},$$

$$Ax = \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t}$$

and we suppose that this limit exists for at least some nonzero $x \in X$.

We consider the abstract reaction–diffusion equation (ARDE) on X

$$u'(t) = Au(t) + f(u(t)), \quad t > 0, u(0) = u_0,$$

where $u : [0, \infty) \rightarrow X$, $f : X \rightarrow X$ and $(A, D(A))$ is the generator of $\{T(t)\}_{t \geq 0}$. This equation has two important particular cases:

- (i) the reaction equation $u'(t) = f(u(t))$, $t > 0$, $u(0) = u_0$;
- (ii) the diffusion equation $u'(t) = Au(t)$, $t > 0$, $u(0) = u_0$.

The following theorem uses results from [4–8], and appears in [2].

Theorem 2.1. Suppose that $f : X \rightarrow X$ is globally Lipschitz (there exists $M > 0$ such that $\|f(x) - f(x')\| \leq M\|x - x'\|$ for all $x, x' \in X$).

- (a) Then (i) has a unique global strong solution $u(t) := S(t)u_0$ for any initial condition $u_0 \in X$ and $\{S(t)\}_{t \geq 0}$ form a semigroup of nonlinear operators, given by

$$u(t) = S(t)u_0 = u_0 + \int_0^t f(u(s)) \, ds.$$

- (b) If A is the generator of the strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X , then (ii) has a unique global mild solution $u(t) = T(t)u_0$ for any initial condition $u_0 \in X$, i.e. u is continuous and satisfies the corresponding integral equation

$$u(t) = u_0 + \int_0^t A(u(s)) \, ds;$$

if $u_0 \in D(A)$, the domain of A , then u is also the unique global strong solution of (ii).

- (c) For any $u_0 \in X$, (ARDE) has a unique global mild solution

$$u(t) := W(t)u_0 = T(t)u_0 + \int_0^t T(t-s)f(u(s)) \, ds$$

that can be computed by the Trotter product formula

$$u(t) = W(t)u_0 = \lim_{n \rightarrow \infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 = \lim_{n \rightarrow \infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0.$$

If $u_0 \in D(A)$ and $f : X \rightarrow X$ is continuously differentiable, then $u(t)$ is the unique global strong solution of (ARDE) and this solution can be computed by the above formula.

Assume that X is an ordered Banach space (a real Banach space endowed with a partial ordering \leq such that: the positive cone X_+ is closed; $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$; $x \geq 0$ implies $\alpha x \geq 0$ for all $x \in X$ and $\alpha \geq 0$; $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$).

Typical examples of ordered Banach spaces are $C_\infty(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$. $C_\infty(\mathbf{R}^n)$ is the space of continuous functions $u : \mathbf{R}^n \rightarrow \mathbf{R}$ such that $u(x) \rightarrow 0$ as $\|x\| \rightarrow \infty$, endowed with the supremum norm and the partial ordering $u \leq v$ whenever $u(x) \leq v(x)$ for all $x \in \mathbf{R}^n$. $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, is endowed with usual norm and the partial ordering $u \leq v$ whenever $u(x) \leq v(x)$ for all $x \in \mathbf{R}^n$ almost everywhere.

Let X be an ordered Banach space. The operator $A : X \rightarrow X$ is called positive if $0 \leq u \leq v$ implies $0 \leq Au \leq Av$. Let A and B be two positive operators. We write $A \leq B$ if $Au \leq Bu$ for all $u \geq 0$.

The following theorem is given from [8,2].

Theorem 2.2. Let X be an ordered Banach space. Assume that the strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ generated by the linear operator A and the nonlinear semigroup $\{S(t)\}_{t \geq 0}$ generated by the Lipschitz continuous function f are positive. If

$$T(t)S(t)u_0 \leq S(t)T(t)u_0$$

holds for all $t \in [0, T]$ and $u_0 \geq 0$, then the unique mild solution $W(t)u_0$ of (ARDE) satisfies

$$\begin{aligned} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 &\leq \left[T\left(\frac{t}{2n}\right) S\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq W(t)u_0 \\ &\leq \left[S\left(\frac{t}{2n}\right) T\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0 \end{aligned}$$

for all $u_0 \geq 0$, $n \in \mathbf{N}$ and $t \in [0, T]$, $T > 0$.

3. Negative definite functions. Feller semigroups

A function $a: \mathbf{R}^n \rightarrow \mathbf{C}$ is said to be *negative definite* if for all $m \in \mathbf{N}$ and $(x^1, x^2, \dots, x^m), x^j \in \mathbf{R}^n$, $1 \leq j \leq m$ and for all m -tuple $(c_1, c_2, \dots, c_m) \in \mathbf{C}^m$ we have

$$\sum_{i,j=1}^m \left[a(x^i) + \overline{a(x^j)} - a(x^i - x^j) \right] c_i \overline{c_j} \geq 0.$$

If $a: \mathbf{R}^n \rightarrow \mathbf{R}$ is a continuous negative definite function, then there exists $C > 0$ such that

$$|a(\xi)| \leq C(1 + |\xi|^2)$$

holds for all $\xi \in \mathbf{R}^n$.

A continuous negative definite function a is described by Lévy–Khinchin formula

$$a(\xi) = c + ib \cdot \xi + q(\xi) + \int_{\mathbf{R}^n \setminus \{0\}} \left[1 - e^{-i\xi \cdot y} - i \frac{\xi \cdot y}{1 + |y|^2} \right] \frac{1 + |y|^2}{|y|^2} d\mu(y)$$

with $c \geq 0$, $b \in \mathbf{R}^n$, q a continuous non-negative definite quadratic form on \mathbf{R}^n and μ a non-negative finite measure on $\mathbf{R}^n \setminus \{0\}$.

In the following, $S(\mathbf{R}^n)$ will be the Schwartz space, i.e. the set of all functions $\varphi \in C^\infty(\mathbf{R}^n)$ such that $\sup_{x \in \mathbf{R}^n} |x^\beta \partial^\alpha \varphi(x)| < \infty$ for all multi-indices α and β . $S(\mathbf{R}^n)$ is dense in $C_\infty(\mathbf{R}^n)$.

The general form of a *pseudodifferential operator* is

$$p(x, D)\varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{ix \cdot \xi} p(x, \xi) \widehat{\varphi}(\xi) d\xi,$$

for $\varphi \in C_0^\infty(\mathbf{R}^n)$, the set of all C^∞ -functions on \mathbf{R}^n with compact support, where

$$\widehat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-ix \cdot \xi} \varphi(x) dx$$

is the Fourier transform. $p(x, \xi)$ is called the *symbol* of the operator $p(x, D)$ (see, for example, [9]).

The *convolution semigroup* on $C_\infty(\mathbf{R}^n)$ generated by a is defined by the formula

$$T(t)u(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} e^{ix \cdot \xi} p_t(\xi) \widehat{u}(\xi) d\xi,$$

for each $t > 0$ and $u \in S(\mathbf{R}^n)$, where

$$p_t(\xi) = e^{-ta(\xi)}.$$

In this case, we observe that for any $t > 0$ the symbol is p_t (note that there is no x -dependence). The function $\xi \rightarrow p_t(\xi)$ is a positive definite function and the infinitesimal generator of $\{T(t)\}_{t \geq 0}$ is

$$Au(x) = -(2\pi)^{-(n/2)} \int_{\mathbf{R}^n} e^{ix \cdot \xi} a(\xi) \widehat{u}(\xi) d\xi,$$

for all $u \in C_0^\infty(\mathbf{R}^n)$, $x \in \mathbf{R}^n$.

Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on $C_\infty(\mathbf{R}^n)$. If $\|T(t)u\| \leq \|u\|$ for all $u \in C_\infty(\mathbf{R}^n)$ and $t \geq 0$, then $\{T(t)\}_{t \geq 0}$ is a contraction semigroup. A strongly continuous positive contraction semigroup on $C_\infty(\mathbf{R}^n)$ is called a *Feller semigroup*. We have an integral representation of the operators which form a Feller semigroup (see [3]).

Theorem 3.1. Let $\{T(t)\}_{t \geq 0}$ be a Feller semigroup on \mathbf{R}^n . For any $t \geq 0$ there exists a unique function

$$p_t : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{C}$$

measurable, locally bounded and such that for any fixed $x \in \mathbf{R}^n$, $\xi \rightarrow p_t(x, \xi)$ is a continuous positive definite function with the property that for any $u \in S(\mathbf{R}^n)$,

$$T(t)u(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} e^{ix \cdot \xi} p_t(x, \xi) \widehat{u}(\xi) d\xi.$$

For $u \in S(\mathbf{R}^n)$, the infinitesimal generator A of $\{T(t)\}_{t \geq 0}$ is

$$Au(x) = (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} e^{ix \cdot \xi} a(x, \xi) \widehat{u}(\xi) d\xi,$$

where

$$a : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{C}, \quad a(x, \xi) = \left. \frac{d}{dt} p_t(x, \xi) \right|_{t=0}.$$

4. Reaction–diffusion equation on $C_\infty(\mathbf{R}^n)$

We consider the generalized reaction–diffusion equation on $C_\infty(\mathbf{R}^n)$

$$\frac{\partial u}{\partial t}(x, t) = (Au(\cdot, t))(x) + \widetilde{f}(x, u(x, t)), \quad u(x, 0) = u_0(x), \quad x \in \mathbf{R}^n,$$

where A is the generator of the Feller semigroup $\{T(t)\}_{t \geq 0}$ on $C_\infty(\mathbf{R}^n)$ and $\widetilde{f} : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$. Re-write this equation in the form of (ARDE), where $[f(u)](x) = \widetilde{f}(u(x))$. Following [2] we suppose that $f : C_\infty(\mathbf{R}^n) \rightarrow C_\infty(\mathbf{R}^n)$ is locally Lipschitz, $f(0) = 0$ and $[f(u)](x) < 0$ for all $u(x) \geq k$, $k \in \mathbf{R}$. Then the differential equation $u'(t) = f(u(t))$, $u(0) = u_0 \geq 0$ has a unique strong global solution given by $u(t) = S(t)u_0$. Thus, we obtain the following theorem (the proof is essentially identical to Theorem 4.3 in [2]).

Theorem 4.1. Let A be the generator of the Feller semigroup $\{T(t)\}_{t \geq 0}$ on $C_\infty(\mathbf{R}^n)$ and let f as above. Then (ARDE) has a unique mild solution $u(t) = W(t)u_0$ for all $u_0 \geq 0$, given by the Trotter product formula

$$u(t) = W(t)u_0 = \lim_{n \rightarrow \infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 = \lim_{n \rightarrow \infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0.$$

We denote by $[\widetilde{S}(t)](u_0(x)) := [S(t)u_0](x)$, where $\widetilde{S}(t)$ is a function that maps the real number $u_0(x)$ to another real number.

Theorem 4.2. We consider verified the above conditions and the following:

- (1) $k_t(x, y) := (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} e^{i(x-y) \cdot \xi} p_t(x, \xi) d\xi$ is bounded for all $x, y \in \mathbf{R}^n$, $t \geq 0$;
 - (2) $S(t)$ and $T(t)$ map $S(\mathbf{R}^n)$ to $S(\mathbf{R}^n)$ for all $t \geq 0$;
 - (3) the function $v \rightarrow [\widetilde{S}(t)](v)$ is concave down on $v > 0$ for any $t > 0$;
 - (4) $[\widetilde{S}(t)][k_t(x, y)u_0(y)] \geq k_t(x, y)\widetilde{S}(t)[u_0(y)]$, for $u_0 \in C_0^\infty(\mathbf{R}^n)$, $u_0 \geq 0$ and all $x, y \in \mathbf{R}^n$, $t \geq 0$.
- Then the following statements are true:

(a) For all $n \in \mathbf{N}$,

$$\begin{aligned} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n u_0 &\leq \left[T\left(\frac{t}{2n}\right) S\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq W(t)u_0 \\ &\leq \left[S\left(\frac{t}{2n}\right) T\left(\frac{t}{2n}\right) \right]^{2n} u_0 \leq \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n u_0. \end{aligned}$$

(b) If $u_0 \in C_0^\infty(\mathbf{R}^n)$, $u_0 \geq 0$ and f is continuously differentiable, then (ARDE) on $C_\infty(\mathbf{R}^n)$ has a unique strong solution $u(t) = W(t)u_0$ given by the Trotter product formula.

Proof. (a) By Jensen's inequality and (1)–(4) we have:

$$\begin{aligned} [S(t) T(t) u_0](x) &= \tilde{S}(t) \left[(2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} p_t(x, \xi) u_0(y) dy \right) d\xi \right] \\ &= \tilde{S}(t) \left[\int_{\mathbf{R}^n} (2\pi)^{-(n/2)} \left(\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} p_t(x, \xi) d\xi \right) u_0(y) dy \right] \\ &= \tilde{S}(t) \left[\int_{\mathbf{R}^n} k_t(x, y) u_0(y) dy \right] \geq \int_{\mathbf{R}^n} \tilde{S}(t) [k_t(x, y) u_0(y)] dy \\ &\geq \int_{\mathbf{R}^n} k_t(x, y) [\tilde{S}(t) u_0](y) dy = \int_{\mathbf{R}^n} k_t(x, y) [S(t) u_0](y) dy \\ &= [T(t) S(t) u_0](x). \end{aligned}$$

Now, the statement is following from [Theorem 2.2](#).

(b) We are applying [Theorems 4.1](#) and [3.1](#). Since u_0 is in the domain of the operator A and f is continuously differentiable, then (ARDE) on $C_\infty(\mathbf{R}^n)$ has a unique strong solution $u(t) = W(t) u_0$ given by the Trotter product formula. \square

Remark 4.3. The existence and uniqueness of the strong solution $u(t) = W(t) u_0$ of (ARDE) on $C_\infty(\mathbf{R}^n)$ yield to solve pointwise the generalized reaction–diffusion equation. In this case, $u(x, t) := [W(t) u_0](x)$ and the convergence from the Trotter product formula is pointwise to $u(x, t)$, uniformly for $x \in \mathbf{R}^n$. Thus, the generalized reaction–diffusion equation in continuous time can be solved numerically by computing solutions to one of its discrete counterparts:

$$u_{n+1}(x) = [T(\tau) S(\tau) u_n](x),$$

where $u_n(x) = u(x, n\tau)$, $\tau = \frac{t}{n}$. We have:

$$\begin{aligned} u_{n+1}(x) &= (2\pi)^{-(n/2)} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} p_t(x, \xi) d\xi \right) [S(\tau) u_n](y) dy \\ &= \int_{\mathbf{R}^n} k_t(x, y) g_\tau(u_n(y)) dy, \end{aligned}$$

where $g_\tau(u_n(y)) = [S(\tau) u_n](y)$. The approximate solutions u_n converge to the unique solution u at any time $t > 0$, when $n \rightarrow \infty$.

Remark 4.4. If $p_t(x, \xi) = e^{-ta(\xi)}$, where a is a continuous negative definite function (see part 3), then we obtain the results from [\[2\]](#).

References

- [1] J. Klafter, I.M. Sokolov, From diffusion to anomalous diffusion: A century after Einstein's Brownian motion, *Chaos* 15 (2005) 26–103.
- [2] B. Baeumer, M. Kovacs, M.M. Meerschaert, Numerical solutions for fractional reaction–diffusion equations, [doi:10.1016/j.camwa.2007.11.012](https://doi.org/10.1016/j.camwa.2007.11.012).
- [3] E. Popescu, A note on Feller semigroups, *Potential Anal.* 14 (2001) 207–209.
- [4] E. Hille, R.S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, Providence, RI, 1974.
- [5] I. Miyadera, S. Ôharu, Approximation of semi-groups of nonlinear operators, *Tôhoku Math. J.* 22 (1970) 22–47.
- [6] H. Brézis, A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, *J. Funct. Anal.* 9 (1972) 63–74.
- [7] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, in: *Applied Mathematical Sciences*, vol. 44, Springer-Verlag, New York, 1983.
- [8] M. Clifft, J.A. Goldstein, M. Wacker, Positivity, Trotter products, and blow-up, *Positivity* 8 (2004) 187–208.
- [9] N. Jacob, *Pseudo-Differential Operators and Markov Processes*, in: *Mathematical Research*, vol. 94, Akademie Verlag, Berlin, 1996.